(Co)homology of iterated semidirect products of finitely generated abelian groups ☆

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Abstract

Let G be a group which admits the structure of an iterated semidirect product of finitely generated abelian groups. We provide a method for constructing a free resolution $\epsilon : X \to \mathbb{Z}$ of the integers over the group ring of G. The heart of this method consists of calculating an explicit contraction from the reduced bar construction of the group ring of G, $\overline{B}(\mathbb{Z}[G])$, to the reduced complex of X. Such contraction is also used as the input datum in the methods described in [7, 8] for calculating a generating set for representative 2-cocycles and n-cocycles over G, respectively. These computations have led to the finding of new cocyclic Hadamard matrices [4]. Finally, some examples and homological calculations were developed, with the aid of *Mathematica*, including dihedral groups.

Key words: Semidirect product of groups, (co)homology of groups, contraction, Homological perturbation theory
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1. Introduction

The homology of a group G is usually determined from a resolution of the integers over the group ring of G, $\epsilon : X \to \mathbb{Z}$. The obtention of "economical" free resolutions for abelian groups (economical in the sense that the requested homological calculations can carried out) is a problem that has been extensively studied [10, 16, 18, 12, 13, 44, 35, 27, 9]. The homological perturbation machinery [39, 11, 22, 23, 24, 29, 36] has supplied a suitable environment to deal with this problem for other discrete groups. For instance, in [37], Rubio described a method to face the problem for computing the homology of central extensions of groups where the main idea has already appeared in [32]. A method based on the same idea has been applied to finitely generated torsion-free nilpotent

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groups [30] and finite p-groups [31, 21]. Using a related perturbation method, Huebschmann attacked, respectively, the problem of the homology computation of finitely generated two-step nilpotent groups [26] and metacyclic groups [28].

Our approach to this problem is similar to [32] and [21], but with the difference that we work directly with reduced complexes (*à la Eilenberg-Mac Lane*) instead of the corresponding full resolutions. This philosophy was first proposed in [18] and has very recently come back from the past with renewed strength, even it has been performed in a computer algebra system for the last decade [15, 38].

The idea of this technique is to construct *homological models* for G, that is, a contraction from the (reduced) bar construction of group ring of G, $\overline{B}(\mathbb{Z}[G])$, to a significantly smaller differential graded module of finite type, hG, so that

$$H_*(G) = H_*(\overline{B}(\mathbb{Z}[G])) = H_*(hG)$$

and the homology of hG may be effectively computed by means of the classical matrix algorithm for computing homology [46]. Examples of homological models for any finitely generated abelian group G were computed in [18].

To construct a resolution of the integers over $\mathbb{Z}[G]$ from a homological model for G it boils down to putting a $\mathbb{Z}[G]$-linear differential on $\mathbb{Z}[G] \otimes hG$ such that an acyclic DG-module results. To this end, we follow these steps:

1. Construct a contraction from $\mathbb{Z}[G] \otimes \overline{B}(\mathbb{Z}[G])$ to $\mathbb{Z}[G] \otimes hG$ using the homological model for G.
2. Perturb the contraction above with the morphism $\theta\cap\colon \mathbb{Z}[G] \otimes \overline{B}(\mathbb{Z}[G]) \to \mathbb{Z}[G] \otimes \overline{B}(\mathbb{Z}[G])$, with $\theta\cap = d - d'$ where d is the differential on the bar resolution, $B(\mathbb{Z}[G])$, and d' is the trivial differential on $\mathbb{Z}[G] \otimes \overline{B}(\mathbb{Z}[G])$.

In the case that the morphism $\theta\cap$ produces a perturbation for the contraction in step 1, then the basic perturbation lemma (BPL) yields a contraction from the bar resolution $B(\mathbb{Z}[G]) = \mathbb{Z}[G] \otimes_\theta \overline{B}(\mathbb{Z}[G])$ to an acyclic DG-module where its underlying graded module structure is $\mathbb{Z}[G] \otimes hG$ and its differential, $d^{\otimes} + d_\theta$, is given by an explicit formula provided by the BPL. We will say that the resolution $\epsilon\colon (\mathbb{Z}[G] \otimes hG,\ d^{\otimes} + d_\theta) \to \mathbb{Z}$ *splits off of the bar resolution.* The authors have already used this argument extensively in [3].

Though there are some papers in the literature concerning the (co)homology of semidirect products of groups (e.g. [43, 14]), we know of none applies to the case of the iterated product of groups that we are dealing with. More concretely, the work in [43] concerns just to the second cohomology groups of semidirect products of finite groups. The work in [14] applies only iterated semidirect products of free groups. In any case, explicit formulas for homological calculation are not provided in any of these two papers. Another remarkable fact is that the homological model for G computed in this paper is suitable for computing a generating set of representative 2-cocycles, and in general, n-cocycles over G (see [7, 8]). These computations have led new cocyclic Hadamard matrices [4] as well as some indispensable cohomological information for further theoretical research in [6]. In addition, our algorithm is a potential source of examples for cocyclic matrices of higher dimensions, which cannot be supplied by the classical methods in the literature, since they only apply to 2-dimensional cocyclic matrices.

In this paper we are concerned with the establishment of explicit formulas for calculating the homology of iterated semidirect products of finitely generated abelian groups (both finite and infinite). It extends the work in [14] to the finite case. Furthermore, our work applies not only to the semidirect products of discrete groups, but also to the semidirect products of simplicial groups. Indeed, the main results of the paper are stated and proved in the framework of simplicial sets.

Our approach is based on four steps. Let H be a simplicial group and K be a simplicial H-group. The first one consists in establishing a simplicial isomorphism (Theorem 1) between the simplicial set $\overline{W}(K \rtimes_\chi H)$, the $\overline{W}$-construction functor applied to the semidirect product $K \rtimes_\chi H$, and the twisted cartesian product $\overline{W}(K) \times_\tau \overline{W}(H)$ relative to the universal twisting function $\tau\colon \overline{W}(H) \to H$ and H-action on $\overline{W}(K)$.

Secondly, the twisted Eilenberg-Zilber Theorem yields a contraction from the normalized chain complex of $\overline{W}(K) \times_\tau \overline{W}(H)$ (we will denote by $C(\overline{W}(K) \times_\tau \overline{W}(H))$) to a twisted tensor product $C(\overline{W}(K)) \otimes_t C(\overline{W}(H))$. Explicit formulas for a contraction of this type are given in [36].

Henceforth we will assume that K and H are ordinary discrete groups. In this particular situation, $C(\overline{W}(K))$, $C(\overline{W}(H))$ and $C(\overline{W}(K \rtimes_\chi H))$ amount to the reduced bar constructions $\overline{B}(\mathbb{Z}[K])$, $\overline{B}(\mathbb{Z}[H])$ and $\overline{B}(\mathbb{Z}[K \rtimes_\chi H])$, respectively. So far, we have in this particular case

- $\overline{B}(\mathbb{Z}[K \rtimes_\chi H]) \overset{\varphi}{\cong} C(\overline{W}(K \rtimes_\chi H)) \overset{\psi}{\cong} C(\overline{W}(K) \times_\tau \overline{W}(H))$.
- $C(\overline{W}(K) \times_\tau \overline{W}(H)) \rightleftharpoons C(\overline{W}(K)) \otimes_t C(\overline{W}(H)) \overset{\varphi^{-1}}{\cong} \overline{B}(\mathbb{Z}[K]) \otimes_t \overline{B}(\mathbb{Z}[H])$.

Next we construct a contraction from $\overline{B}(\mathbb{Z}[K]) \otimes_t \overline{B}(\mathbb{Z}[H])$ to a significantly smaller free DG-module of finite type, hKH. This last object is a certain twisted tensor product $hK \otimes hH$ of small DG-modules hK and hH onto which $\overline{B}(\mathbb{Z}[K])$ and $\overline{B}(\mathbb{Z}[H])$ contract respectively. In the case that K and H are finitely generated abelian groups, such explicit contractions to hK and hH exist [18]. The key point is to guarantee the convergence of the related perturbation process (Theorem 6). From these data and the work of the authors in [3], it is easy to derive at once a small free resolution of the ground ring over $\mathbb{Z}[K \rtimes_\chi H]$ (Theorem 7). This amounts to putting a $\mathbb{Z}[K \rtimes_\chi H]$-linear differential on $\mathbb{Z}[K \rtimes_\chi H] \otimes hKH$ such that an acyclic chain complex results. Furthermore, we prove that such a resolution is endowed an A_∞-coalgebra structure, naturally inherited from the reduced complex of the bar resolution (Theorem 13).

In the fourth step, the tensor product of contractions supplies a homological model for iterated semidirect products of the class of groups under consideration (Theorem 10), hence the correspondent resolutions (Theorem 11).

From these resolutions, cohomology may be easily determined. It suffices to apply the *Hom* functor to the homological model just described. Further details can be found in [21, 7, 8].

Furthermore, the method fits over other semidirect products of groups (as well as iterated products of groups), even though the fibre groups K may not be a finitely generated abelian group (see Remark 2 and Remark 4). An extended version of the method for iterated products of central extensions and semidirect products of finitely generated abelian groups has been implemented in *Mathematica* by the authors (see [2, 5]). Some calculations with this package have led to find new cocyclic Hadamard matrices [1, 4, 7].

We organize the paper as follows. Section 2 is devoted describing a (co)homological model for a semidirect product of finite generated abelian groups. For the sake of clarity it begins by introducing some notations and results on Simplicial Topology and Homological Algebra. In Section 3, a homological model for iterated semidirect products of finite generated abelian groups is determined. In fact, Theorems 6 and 10 give rise to explicit (not recursive) formulas for calculating the homology of such a group. Section 4 is devoted showing some examples and calculations, with the aid of our *Mathematica* package, including the cases of finite and infinite dihedral groups. In section 5 we include some comments about the simplification of the formulas that our method provides. In particular, we analyze the algebra structures beneath the resolution just constructed, which might lead to improvements in the homological calculations in the near future.

2. Describing a (co)homological model for $K \rtimes_\chi H$

In this section, we describe a homological model for a single semidirect product $K \rtimes_\chi H$ of (discrete) finitely generated abelian groups K and H.

Firstly, we recall the definition of semidirect product of two groups H and K. Let χ be an action of H on K, i.e. $\chi : H \times K \to K$ with $\chi(h,k) = \alpha(h)(k)$ where $\alpha : H \to \mathrm{Aut}(K)$ is a homomorphism. The *semidirect product* of H and K with respect to χ, $K \rtimes_\chi H$ (or $K \rtimes_\alpha H$), is the set $K \times H$, endowed with the group law

$$(k,h) \cdot (k',h') = (k + \chi(h,k'), h + h').$$

We will write hk instead of $\chi(h,k)$ when no confusion can arise.

Example 1. The dihedral group

$$D_{2m} =< h,k : \; h^2 = 1, k^m = 1, hkh^{-1} = k^{-1} >$$

is the semidirect product $\mathbb{Z}_m \rtimes_\chi \mathbb{Z}_2$, $m \geq 2$, for $\chi(0,k) = k$, $\chi(1,k) = -k$.

In order to describe a homological model for $K \rtimes_\chi H$, we need to work in the framework of simplicial sets and use the techniques that the homological perturbation theory provides. We recall some basic concepts of Simplicial Topology and Homological Algebra. More details can be found in [34] and in [33] respectively.

A *simplicial group* G is a simplicial set $G = (G_n, \partial_i, s_i)$ where every G_n is a group and every face or degeneracy operator is compatible with the group structures. If G has only one 0-simplex, then G is called *reduced*.

The $\overline{W}$*-construction* (or the *classifying construction* $(\overline{W})$) for a simplicial group G, denotes by $\overline{W}(G)$, is a new simplicial set defined as follows:

$$
\begin{array}{rcl}
\overline{W}_0(G) & = & \{[\,]\}; \\
\overline{W}_n(G) & = & G_{n-1} \times \cdots \times G_0, \quad n > 0; \\
s_0[\,] & = & [1]; \\
\partial_i[g_0] & = & [\,], \quad i = 0, 1; \\
\partial_0[g_n, \ldots, g_0] & = & [g_{n-1}, \ldots, g_0], \\
\partial_{i+1}[g_n, \ldots, g_0] & = & [\partial_i g_n, \ldots, \partial_1 g_{n-i+1}, g_{n-i-1}\partial_0 g_{n-i}, g_{n-i-2}, \ldots, g_0], \\
s_0[g_{n-1}, \ldots, g_0] & = & [1, g_{n-1}, \ldots, g_0], \\
s_{i+1}[g_n, \ldots, g_0] & = & [s_i g_n, \ldots, s_0 g_{n-i}, 1, g_{n-i-1}, \ldots, g_0];
\end{array}
$$

where $[\,]$ denotes the unique element of $\overline{W}_0(G)$, 1 denotes the identity elements of G (at each simplicial degree) and $[g_{n-1}, \ldots, g_0]$ denotes a generic element of $\overline{W}_n(G)$, for $n > 0$. $\overline{W}(G)$ is also called *a classifying space* for G.

If G is an ordinary discrete group then $\overline{W}(G) = \overline{W}({}^sG)$, for ${}^sG_m = G$, $\forall m \geq 0$, and all face and degeneracy operators are the identity maps. For clarity in the exposition, we denote ${}^sG$ simply by G itself in the sequel.

We need here the *reduced bar construction* $\overline{B}(A)$ of a DGA-algebra A. Recall that it is defined as the connected DGA-coalgebra, $\overline{B}(A) = T^c(s(\bar{A})))$, where $T^c(\;)$ is the tensor coalgebra, $s(\;)$ is the suspension functor and $\bar{A}$ is the augmentation ideal of A. The element of $\overline{B}_0(A)$ corresponding to the identity element of Λ (ground ring) is denoted by $[\,]$ and the element $s\bar{a_1} \otimes \cdots \otimes s\bar{a_n}$ of $\overline{B}(A)$ is denoted by $[a_1|\cdots|a_n]$. The tensor and simplicial degrees of the element $[a_1|\cdots|a_n]$ are $|[a_1|\cdots|a_n]|_t = \sum |a_i|$ and $|[a_1|\cdots|a_n]|_s = n$, respectively; its total degree is the sum of its tensor and simplicial degree. The tensor and simplicial differential are defined by:

$$d_t([a_1|\cdots|a_n]) \;=\; -\textstyle\sum_i (-1)^{e_{i-1}}[a_1|\cdots|d_A(a_i)|\cdots|a_n],$$

and

$$d_s([a_1|\cdots|a_n]) \;=\; \textstyle\sum_i (-1)^{e_i}[a_1|\cdots|\mu_A(a_i \otimes a_{i+1})|\cdots|a_n]$$

where

$$e_i = i + |a_1| + \cdots + |a_i|.$$

If the product of A is commutative, a product $*$ (called shuffle product) can be defined on $\overline{B}(A)$. For every discrete group G, $\overline{B}(\mathbb{Z}[G])$ amounts to $C(\overline{W}(G))$ by means of the following isomorphism

$$\varphi\colon \overline{B}(\mathbb{Z}[G]) \to C(\overline{W}(G)),$$

$$\varphi([g_0|\dots|g_n]) = \begin{cases} (g_0,\dots,g_n), & G \text{ is abelian} \\ (-1)^{\lceil \frac{n+1}{2} \rceil +1}(g_n,\dots,g_0), & \text{Otherwise.} \end{cases}$$

Consider two simplicial sets F, B and a simplicial group G which operates on F from the left. A *twisted cartesian product* E with fibre F, base B and structural group G consists of a simplicial set $E_n = F_n \times B_n$ and

$$\begin{array}{rcl} \partial_0(a,b) & = & (\tau b \star \partial_0 a, \partial_0 b) \\ \partial_i(a,b) & = & (\partial_i a, \partial_i b), \quad \text{for } \ i>0 \\ s_i(a,b) & = & (s_i a, s_i b), \quad \text{for } \ i \geq 0; \end{array}$$

as face and degeneracy operators. Here $\star\colon G\times F \to F$ is the action of G on F and τ is a *twisting function*, i.e., $\tau_n\colon B_n \to G_{n-1},\ n\geq 1$ satisfies

$$\begin{array}{rcl} \partial_0\tau(b) & = & [\tau(\partial_0 b)]^{-1}\cdot\tau(\partial_1 b) \\ \partial_i\tau(b) & = & \tau(\partial_{i+1}b), \quad \text{for } \ i>0 \\ s_i\tau(b) & = & \tau(s_{i+1}b), \quad \text{for } i\geq 0 \\ \tau(s_0 b) & = & 1, \end{array}$$

where 1 denotes the identity element of the corresponding group G_n. We write $E = F\times_\tau B$.

Example 2. Let K and H be two simplicial groups where H operates on K from the left, then a TCP $\overline{W}(K)\times_\tau \overline{W}(H)$ with fibre $\overline{W}(K)$, base $\overline{W}(H)$ and structural group H can be defined via the action

$$\begin{array}{rcl} \star\colon H\times\overline{W}(K) & \longrightarrow & \overline{W}(K) \\ (h,[k_{n-1},\dots,k_0]) & \longrightarrow & [h\cdot k_{n-1},\dots,h\cdot k_0]; \end{array}$$

and twisting function $\tau_n : \overline{W}_n(H) \longrightarrow H_{n-1}$,

$$\tau_n[h_{n-1},\dots,h_0] = h_{n-1}.$$

Theorem 1. *In the conditions of the example above, there is an explicit simplicial isomorphism*

$$\psi : \overline{W}(K \rtimes_\chi H) \longrightarrow \overline{W}(K)\times_\tau \overline{W}(H).$$

PROOF. Define ψ and ψ^{-1} to be

$$\psi_n[(k_{n-1},h_{n-1}),\dots,(k_0,h_0)] =$$
$$([h_{n-1}^{-1}\cdot k_{n-1},\dots,\partial_0^{i-1}h_{n-1}^{-1}\dots\partial_0 h_{n-i+1}^{-1}\, h_{n-i}^{-1}\cdot k_{n-i},\dots,\partial_0^{n-1}h_{n-1}^{-1}\dots\partial_0 h_1^{-1}\, h_0^{-1}\cdot a_0],$$
$$[h_{n-1},\dots,h_0]);$$

$$\psi_n^{-1}([k_{n-1},\dots,k_0],\,[h_{n-1},\dots,h_0]) =$$
$$[(h_{n-1}k_{n-1},h_{n-1})\dots,(h_{n-i}\partial_0 h_{n-i+1}\dots\partial_0^{n-i+1}h_{n-1}\cdot k_{n-i},h_{n-i}),\dots,$$
$$(h_0\partial_0 h_1\dots\partial_0^{n-1}h_{n-1}\cdot k_0,h_0)].$$

Now the statement of the theorem follows by direct inspection. The proof is left to the reader. □

Now, we make a precise definition of the objects studied in the homological perturbation theory and sketch a familiar example.

Let N and M be two DG-modules. Their differentials will be denoted respectively by d_N and d_M or simply by d when no confusion can arise. $d^{\otimes}$ denotes the trivial differential, $d_N \otimes 1 + 1 \otimes d_M$, on $N \otimes M$. A *contraction* (see [17], [29]) is a data set $c : \{N, M, f, g, \phi\}$ where $f : N \to M$ and $g : M \to N$ are morphisms of DG-modules (called, respectively, *the projection* and *the inclusion*) and $\phi : N \to N$ is a morphism of graded modules of degree +1 (called *the homotopy operator*). These data are required to satisfy the rules: **(c1)** $fg = 1_M$, **(c2)** $\phi d_N + d_N \phi + gf = 1_N$ **(c3)** $\phi\phi = 0$, **(c4)** $\phi g = 0$ and **(c5)** $f\phi = 0$. These last three are called the side conditions [32]. In fact, these may always be assumed to hold, since the homotopy ϕ can be altered to satisfy these conditions [23]. These formulas imply that both chain complexes N and M have the same homology. We will also denote a contraction c by either ${}^{\phi:}N \underset{g}{\overset{f}{\rightleftharpoons}} M$ or $N \Rightarrow M$.

If we have two contractions (f_i, g_i, ϕ_i) from N_i to M_i, for $i = 1, 2$ then, the following contractions can be constructed (see [17]):

- The tensor product contraction $(f_2 \otimes f_1,\ g_1 \otimes g_2,\ \phi_1 \otimes g_2 f_2 + 1_{N_1} \otimes \phi_2)$ from $N_1 \otimes N_2$ to $M_1 \otimes M_2$.
- If $N_2 = M_1$, the composition contraction $(f_2 f_1,\ g_1 g_2,\ \phi_1 + g_1 \phi_2 f_1)$ from N_1 to M_2.

The Eilenberg-Zilber theorem [19] provides the most classic example of a contraction of chain complexes.

An Eilenberg-Zilber contraction is defined in [18] by the data set

$$^{SHI:}C(F \times B) \underset{EML}{\overset{AW}{\rightleftharpoons}} C(F) \otimes C(B)$$

where F and B are simplicial sets. Here $C(F)$ denotes the normalized chain complex associated to a simplicial set F with coefficients in $\mathbb{Z}$. The Alexander-Whitney operator $AW\colon C(F \times B) \to C(F) \otimes C(B)$, the Eilenberg-Zilber operator $EML\colon C(F) \otimes C(B) \to C(F \times B)$ and the Shih operator (of degree +1) $SHI\colon C(F \times B) \to C(F \times B)$ are defined by the following formulas:

$$\begin{aligned}
AW(a_n \times b_n) &= \sum_{i=0}^{n} \partial_{i+1} \cdots \partial_n a_n \otimes \partial_0 \cdots \partial_{i-1} b_n, \\
EML(a_p \otimes b_q) &= \sum_{(\alpha,\beta)\in\{(p,q)-shuffles\}} (-1)^{sg(\alpha,\beta)} (s_{\beta_q} \cdots s_{\beta_1} a_p \times s_{\alpha_p} \cdots s_{\alpha_1} b_q), \\
SHI(a_n \times b_n) &= \\
&= \sum (-1)^{m+sg(\alpha,\beta)} (s_{\beta_q+m} \cdots s_{\beta_1+m} s_{m-1} \partial_{n-q+1} \cdots \partial_n a_n \times \\
&\qquad \times s_{\alpha_{p+1}+m} \cdots s_{\alpha_1+m} \partial_m \cdots \partial_{m+p-1} b_n);
\end{aligned}$$

the last sum is taken over the indices $0 \le q \le n-1$, $0 \le p \le n-q-1$ and and $(\alpha, \beta) \in \{(p+1, q)\text{-shuffles}\}$ where $m = n-p-q$ and $sg(\alpha, \beta) =$

$\sum_{i=1}^{p+1}(\alpha_i - (i-1))$. We define AW, EML and SHI to be the 1, 1 and 0 maps in degree 0, respectively.

One of the cornerstones of the homological perturbation theory is the Basic Perturbation Lemma. It provides a beautiful way of unifying many disparate results in Algebraic Topology concerning chain homotopy equivalences, and it can be used to find new results as well.

Now, we recall the concept of a perturbation datum. Let N be a graded module and let $f : N \to N$ be a morphism of graded modules. The morphism f is *pointwise nilpotent* if for all $x \in N$ $(x \neq 0)$, a positive integer n exists (in general, the number n depends on the element x) such that $f^n(x) = 0$. A *perturbation of a DG-module* N is a morphism of graded modules $\delta : N \to N$ of degree -1, such that $(d_N + \delta)^2 = 0$ and $\delta_1 = 0$, i.e. $d_N + \delta$ is a new differential on N. A *perturbation datum of the contraction* $c : \{N, M, f, g, \phi\}$ is a perturbation δ of the DGA-module N verifying that the composition $\phi\delta$ is pointwise nilpotent.

A *Transference Problem* consists of a contraction c: $\{M,\ N,\ f,\ g,\ \phi\}$ together with a perturbation δ of the DG-module N. The problem is to determine new morphisms $d_\delta,\ f_\delta,\ g_\delta$ and ϕ_δ such that $c_\delta : \{(N, d_N{+}\delta),\ (M, d_M{+}d_\delta),\ \ f_\delta,\ g_\delta,\ \phi_\delta\}$ is a contraction.

The Basic Perturbation Lemma ([11, 23, 24, 36]) gives an explicit solution to the Transference Problem, assuming that δ is a perturbation datum of c.

Theorem 2. *(***BPL***)*

Let $c : \{N, M, f, g, \phi\}$ be a contraction and $\delta : N \to N$ a perturbation datum of c. Then, a new contraction

$$c_\delta : \{(N, d_N + \delta),\ (M, d_M + d_\delta),\ \ f_\delta,\ g_\delta,\ \phi_\delta\}$$

is defined by the formulas: $d_\delta = f\delta\Sigma_c^\delta g$; $f_\delta = f(1 - \delta\Sigma_c^\delta\phi)$; $g_\delta = \Sigma_c^\delta g$; $\phi_\delta = \Sigma_c^\delta\phi$; where

$$\Sigma_c^\delta \ = \ \sum_{i\geq 0}(-1)^i\ (\phi\delta)^i \ = \ 1 - \phi\delta + \phi\delta\phi\delta - \cdots + (-1)^i(\phi\delta)^i + \cdots .$$

Let us note that $\Sigma_c^\delta(x)$ is a finite sum for each $x \in N$, because of the pointwise nilpotency of the composition $\phi\delta$. Moreover, it is obvious that the morphism d_δ is a perturbation of the DG-module (M, d_M).

The twisted Eilenberg-Zilber theorem can be seen as an important example of the usefulness of this lemma (see [39]). It solves the Transference Problem for twisted cartesian products.

Theorem 3. (Twisted Eilenberg-Zilber Theorem) *[39, 22]*
Let $F \times_\tau B$ be the TCP with fibre F, base B and structural group G. Then, the morphism

$$\delta(a, b) = (\tau b \star \partial_0 a, \partial_0 b) - (\partial_0 a, \partial_0 b), \quad (a, b) \in C_N(F \times B)$$

is a perturbation datum of the contraction,

$$^{SHI:}C(F\times B)\underset{EML}{\overset{AW}{\rightleftharpoons}}C(F)\otimes C(B).$$

From these data a new contraction (called the twisted Eilenberg-Zilber contraction*) is obtained by applying BPL:*

$$^{SHI_\delta:}C(F\times_\tau B)\underset{EML_\delta}{\overset{AW_\delta}{\rightleftharpoons}}C(F)\otimes_t C(B)$$

where the bigger chain complex is associated to $F\times_\tau B$, and the smaller one consists of a twisted tensor product along the twisting cochain t, for $t=p\circ d_\delta\circ\rho$

$$C(B)\xrightarrow{\rho}C(G)\otimes C(B)\xrightarrow{d_\delta}C(G)\otimes C(B)\xrightarrow{p}C_N(G) \tag{1}$$

where

$\rho(x)=1_0\otimes x,\quad 1_0$ *being the identity element of G_0 and* $p(y\otimes x)=\begin{cases}0, & x\notin B_0\\ y, & x\in B_0\end{cases}$

So that, $C(F)\otimes_t C(B)$ is a differential graded module whose underlying module structure is given by the ordinary tensor product $C(F)\otimes C(B)$ and whose differential is given by $d^{\otimes}+t\cap$, where $d^{\otimes}=d\otimes 1+1\otimes d$ and $t\cap$ is given by:

$$t\cap=(\mu_{C(F)}\otimes 1)(1\otimes t\otimes 1)(1\otimes\Delta_{C(B)}), \tag{2}$$

where $\mu_{C(F)}$ is the module action induced by the the action $\star\colon G\times F\to F$. Hence,

$$d_\delta=t\cap.$$

Applying the above theorem to $\overline{W}(K)\times_\tau\overline{W}(H)$, the TCP defined in Example 2, it follows

$$^{SHI_\delta:}C(\overline{W}(K)\times_\tau\overline{W}(H))\underset{EML_\delta}{\overset{AW_\delta}{\rightleftharpoons}}C(\overline{W}(K))\otimes_t C(\overline{W}(H)). \tag{3}$$

Furthermore, if K and H are ordinary discrete groups we will give an explicit formula for the twisting cochain t and for the morphism $t\cap$ (see lemmas 4 and 5).

To summ up, given that the semidirect product $K\rtimes_\chi H$ where K and H are simplicial groups with H operating on K from the left, we have

1. $C(\overline{W}(K\rtimes_\chi H))\overset{\psi}{\cong}C(\overline{W}(K)\times_\tau\overline{W}(H))$ (by Theorem 1).

2. $^{SHI_\delta:}C(\overline{W}(K)\times_\tau\overline{W}(H))\underset{EML_\delta}{\overset{AW_\delta}{\rightleftharpoons}}C(\overline{W}(K))\otimes_t C(\overline{W}(H))$ (by Theorem 3).

From now on, we will assume that K and H are ordinary discrete groups, unless otherwise stated.

Lemma 4. *An explicit formula for the twisting cochain* $t : C(\overline{W}(H)) \to C(H)$ *is given by*

$$t(h_{n-1}, \ldots, h_0) = \begin{cases} h_0 - 1, & \text{if } n = 1, \\ 0, & \text{if } n \geq 2. \end{cases}$$

Proof. Attending to Theorem 3 applied to the TCP $\overline{W}(K) \times_\tau \overline{W}(H))$ (see Example 2), the twisting cochain $t : C(\overline{W}(H)) \to C(H)$ is given by the composition $t = p\, d_\delta\, \rho$,

$$C(\overline{W}(H)) \xrightarrow{\rho} C(H) \otimes_t C(\overline{W}(H)) \xrightarrow{d_\delta} C(H) \otimes_t C(\overline{W}(H)) \xrightarrow{p} C(H),$$

where $\rho(h_{n-1}, \ldots, h_0) = 1 \otimes (h_{n-1}, \ldots, h_0)$, $p(h \otimes ()) = h$ (zero otherwise) and $d_\delta = AW\delta \sum_{i\geq 0}(-1)^i (SHI\delta)^i EML$ is the perturbation datum provided by BPL when

$$^{SHI:} C(H \times \overline{W}(H)) \underset{EML}{\overset{AW}{\rightleftarrows}} C(H) \otimes C(\overline{W}(H))$$

is perturbed by means of

$$\delta(h, (h_{n-1}, \ldots, h_0)) = (h_{n-1} \cdot h, (h_{n-2}, \ldots, h_0)) - (h, (h_{n-2}, \ldots, h_0)).$$

It is readily checked that the composition $\delta\, EML\, \rho$ consists of

$$\begin{array}{lcl} (h_{n-1}, \ldots, h_0) & \xrightarrow{\rho} & 1 \otimes (h_{n-1}, \ldots, h_0) \\ & \xrightarrow{EML} & (1, (h_{n-1}, \ldots, h_0)) \\ & \xrightarrow{\delta} & (h_{n-1}, (h_{n-2}, \ldots, h_0)) - (1, (h_{n-2}, \ldots, h_0)). \end{array}$$

Independent of the value of n, the application of SHI to the output above is always null. This is obvious for $n = 1$, since SHI is defined as the zero map acting on simplicial degree 0. For $n \geq 2$, the SHI map introduces some degeneracy operators s_j on the term in $\overline{W}(H)$, so that the final output in $C(H \times \overline{W}(H))$ is the image of the degeneracy operator (s_j, s_j), and hence zero (notice that H denotes here the simplicial version of the discrete group H, whose degeneracy and face operators are the identity map on H).

This way, the composition $p\, d_\delta\, \rho$ reduces to $p\, AW\delta EML\, i$.

All summands of

$$AW((h, (h_{n-2}, \ldots, h_0)) = \sum_{i=0}^{n-1} \partial_{n-i}^{n-i} h \otimes \partial_0^i (h_{n-2}, \ldots, h_0)$$

are zero but the one correspondent to $i = 0$, so that the element in $C(H)$ is located at simplicial degree 0 (and hence is not degenerated). Thus,
$AW\delta EML\, i(h_{n-1}, \ldots, h_0) = (h_{n-1}, (h_{n-2}, \ldots, h_0)) - (1, (h_{n-2}, \ldots, h_0))$.

Taking into account that the projection p is null acting on the elements of $C(\overline{W}(H))$ of simplicial degree greater than 0, we finally conclude that

$$t(h_{n-1},\dots,h_0) = p\; d_\delta\; i(h_{n-1},\dots,h_0) = \begin{cases} h_0-1, & n=1,\\ 0, & n\geq 2. \end{cases} \qquad \square$$

Remark 1. If the basis group H of the semidirect product is located on the left-hand side, $H_\chi \ltimes K$, the precedent twisting cochain t must be changed in turn to the opposite $t'=-t$.

Lemma 5. *An explicit formula for the morphism*

$$t\cap : C(\overline{W}(K))\otimes C(\overline{W}(H)) \to C(\overline{W}(K))\otimes C(\overline{W}(H))$$

is given by

$$t\cap((k_{n-1},\dots,k_0)\otimes(h_{m-1},\dots,h_0)) =$$
$$(-1)^n((h_{m-1}k_{n-1},\dots,h_{m-1}k_0)\otimes(h_{m-2},\dots,h_0)-(k_{n-1},\dots,k_0)\otimes(h_{m-2},\dots,h_0))$$

Proof. It is a simple inspection. (see the formula for $t\cap$ given in (2)). $\square$

Our next goal will be to construct a contraction from $C(\overline{W}(K))\otimes_t C(\overline{W}(H))$ to hKH (a DG-module of finite type). To this end, we assume homological models

$$C(\overline{W}(K)) \overset{\varphi^{-1}}{\cong} {}^{\phi_K}\!:\overline{B}(\mathbb{Z}[K]) \underset{g_K}{\overset{f_K}{\rightleftarrows}} hK \qquad \text{and} \qquad C(\overline{W}(H)) \overset{\varphi^{-1}}{\cong} {}^{\phi_H}\!:\overline{B}(\mathbb{Z}[H]) \underset{g_H}{\overset{f_H}{\rightleftarrows}} hH$$

for K and H are known, respectively. For instance, if K and H are (discrete) finite generated abelian groups, such homological models exist [17, 18], in terms of certain products of some exterior and divided power algebras $E(u)$ and $\Gamma(v)$. With these homological models at hand we construct

$$^{1\otimes\phi_H+\phi_K\otimes g_H f_H}\!: C(\overline{W}(K))\otimes C(\overline{W}(H)) \underset{g_K\otimes g_H}{\overset{f_K\otimes f_H}{\rightleftarrows}} hK\otimes hH.$$

If the morphism $t\cap$ (ver lemma 5) is a perturbation datum of the contraction above, then the BPL yields the desired contraction.

Now, we recall the homological models for finite generated abelian groups. Here $E(u)$ denotes the free DGA-algebra endowed with trivial differential and generators 1 (at degree 0) and u (at degree 1), so that $u\cdot u=0$. And $\Gamma(v)$ denotes the free DGA-algebra endowed with trivial differential and generators $\gamma_k(v)$ (at degree $2k$, $k\geq 0$, $\gamma_0(v)=1$), such that $\gamma_k(v)\gamma_h(v)=\frac{(k+h)!}{k!h!}\gamma_{k+h}(v)$.

The comparison theorem for resolutions provides a homological model for $\mathbb{Z}$ (see [3] for details), ${}^{\phi_{\mathbb{Z}}}\!:\overline{B}(\mathbb{Z}[\mathbb{Z}]) \underset{g_{\mathbb{Z}}}{\overset{f_{\mathbb{Z}}}{\rightleftarrows}} E(u)$, which is a subtle modification of that in [18] (they only differ in the homotopy operator $\phi_{\mathbb{Z}}$).

Here $g_{\mathbf{Z}}(u) = [1]$, $f_{\mathbf{Z}}([n_1|\cdots|n_q]) = \begin{cases} n_1\, u, & \text{if } q = 1 \\ 0, & \text{if } q > 1 \end{cases}$ and

$$\phi_{\mathbf{Z}}[n_1|\dots|n_k] = \begin{cases} (-1)^k \displaystyle\sum_{i=1}^{n_k-1} [n_1|\dots|n_{k-1}|i|1], & \text{if } n_k > 0, \\ 0, & \text{if } n_k = 0, \\ (-1)^{k+1} \displaystyle\sum_{i=1}^{|n_k|} [n_1|\dots|n_{k-1}|-i|1], & \text{if } n_k < 0. \end{cases} \quad (4)$$

In [18] a homological model ${}^{\phi_{\mathbf{Z}_n}}\!:\!\overline{B}(\mathbb{Z}[\mathbb{Z}_n]) \underset{g_{\mathbf{Z}_n}}{\overset{f_{\mathbf{Z}_n}}{\rightleftharpoons}} (E(u) \otimes \Gamma(v), d)$ for $\mathbb{Z}_n$ is also described, such that $d(u) = 0$, $d(v) = n \cdot u$ and $g_{\mathbf{Z}_n}(u) = [1]$,

$$g_{\mathbf{Z}_n}(\gamma_k(v)) = \sum_{x_i \in \mathbf{Z}_n} [1|x_1|\cdots|1|x_k], \qquad g_{\mathbf{Z}_n}(u\gamma_k(v)) = \sum_{x_i \in \mathbf{Z}_n} [1|x_1|\cdots|1|x_k|1],$$

$$\begin{aligned} f_{\mathbf{Z}_n}[x_1|y_1|\cdots|x_m|y_m] &= [\prod_{i=1}^{m} \delta_{x_i,y_i}]\gamma_m(v), \\ f_{\mathbf{Z}_n}[x_1|y_1|\cdots|x_m|y_m|z] &= [z \prod_{i=1}^{m} \delta_{x_i,y_i}]u\gamma_m(v), \end{aligned} \quad \text{for } \delta_{x_i,y_i} = \begin{cases} 0, & x_i + y_i < n, \\ 1, & x_i + y_i \geq n, \end{cases}$$

and $\phi_{\mathbf{Z}_n}([x_1|\cdots|x_k]) = -\varphi_{\mathbf{Z}_n}([x_1|\cdots|x_k])$, for $\varphi_{\mathbf{Z}_n}[\;] = 0$, $\varphi_{\mathbf{Z}_n}[x] = \sum_{i=1}^{x-1}[1|i]$, and

$$\varphi_{\mathbf{Z}_n}[x|y|\sigma] = \sum_{i=1}^{x-1}[1|i|y|\sigma] + \delta_{x,y}\sum_{k=1}^{n-1}[1|k|\varphi_{\mathbf{Z}_n}\sigma]. \quad (5)$$

It is well-known that if A is a finitely generated abelian group then A can be written in the form $A = \mathbb{Z}^m \times \mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_n}$, where each l_i denotes a power of a prime. From the data above, a homological model for such an abelian group A may be constructed in a straightforward manner [18], by simply applying $n + m$ times the Eilenberg-Zilber theorem, and tensoring up the $n + m$ correspondent single homological models, so that the following chain of contractions is obtained:

$$\begin{gathered} C(\overline{W}(\mathbb{Z}[A])) \cong C(\overline{W}(\mathbb{Z}[\mathbb{Z}]) \times \overset{m}{\dots} \times \overline{W}(\mathbb{Z}[\mathbb{Z}]) \times \overline{W}(\mathbb{Z}[\mathbb{Z}_{l_1}]) \times \ldots \times \overline{W}(\mathbb{Z}[\mathbb{Z}_{l_n}])) \\ {\scriptstyle (AW_{\otimes},\, EML_{\otimes},\, SHI_{\otimes})} \Downarrow \\ C(\overline{W}(\mathbb{Z}[\mathbb{Z}])) \otimes \ldots \otimes C(\overline{W}(\mathbb{Z}[\mathbb{Z}_{l_n}])) \\ {\scriptstyle (f_{\otimes},\, g_{\otimes},\, \phi_{\otimes})} \Downarrow \\ E_1 \otimes \ldots \otimes E_m \otimes (E_{m+1} \otimes \Gamma_1) \otimes \ldots \otimes (E_{m+n} \otimes \Gamma_n). \quad (6) \end{gathered}$$

Now, we have all the necessary elements to state the following result.

Theorem 6. *Let K and H be finitely generated abelian groups, and let $K \rtimes_\chi H$ be the semidirect product of H and K with respect to the action χ. Then, the morphism $t\cap$ (lemma 5) is a perturbation datum of*

$$ {}^{1\otimes\phi_H+\phi_K\otimes g_H f_H} : C(\overline{W}(K)) \otimes C(\overline{W}(H)) \underset{g_K\otimes g_H}{\overset{f_K\otimes f_H}{\rightleftarrows}} hK \otimes hH, \tag{7} $$

and hence a homological model for $K \rtimes_\chi H$ is completely determined.

Proof. Obviously, $t\cap$ is a perturbation of the complex $C(\overline{W}(K))\otimes C(\overline{W}(H))$, so if we prove that $(1\otimes\phi_H+\phi_K\otimes g_H f_H)t\cap$ is pointwise nilpotent then $t\cap$ will be a perturbation of the contraction (7)

To this end, we look for a filtration $\{D_q\}_{q\geq 0}$ on $C(\overline{W}(K))\otimes C(\overline{W}(H))$, such that $t\cap$ reduces the filtration degree, as $(1\otimes\phi_H+\phi_K\otimes g_H f_H)$ preserves the filtration degree. Consequently, the composition $(1\otimes\phi_H+\phi_K\otimes g_H f_H)t\cap$ reduces the filtration degree, and is shown to be pointwise nilpotent.

Assume that $H = \mathbb{Z}^m \times \mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_n}$. We define $F_q(C(W(H))$ to be the sub-DG-module generated by those tuples $(x^1_{t-1}\times\cdots\times x^{m+n}_{t-1},\cdots,x^1_0\times\cdots\times x^{m+n}_0)$ such that $\sum_{i,j}|x^j_i|\leq q$. We define the filtration $\{D_q\}_{q\geq 0}$ so that

$$ D_q = C(\overline{W}(K)) \otimes F_q(C(\overline{W}(H)). $$

Taking into account formulas (4), (5) and (6), it is readily checked that the homotopy operator ϕ_H and the composition $g_H f_H$ preserve the filtration degree. Furthermore, using the formula giving in Lemma 5 and by a simple inspection, we can state that $t\cap$ decreases the filtration degree, at least in one degree. This concludes the proof.

Hence, BPL gives rise to the contraction

$$ {}^{(1\otimes\phi_H+\phi_K\otimes g_H f_H)_{t\cap}} : C(\overline{W}(K))\otimes_t C(\overline{W}(H)) \underset{(g_K\otimes g_H)_{t\cap}}{\overset{(f_K\otimes f_H)_{t\cap}}{\rightleftarrows}} (hK\otimes hH, 1\otimes d + d\otimes 1 + d_{t\cap}). $$

For the sake of simplicity, we note $\phi_t = (1\otimes\phi_H+\phi_K\otimes g_H f_H)_{t\cap}$, $f_t = (f_K\otimes f_H)_{t\cap}$ and $g_t = (g_K\otimes g_H)_{t\cap}$. $\square$

Remark 2. Notice that the proof of the theorem above works on any semidirect product $K \rtimes_\chi H$, for H a finitely generated abelian group, and for K a group with a known homolological model, but not necessarily abelian. We rely on this fact to extend the above theorem to iterated semidirect products in the next section.

To summarize, under the hypothesis of Theorem 6 we can link the next complexes

$$ \begin{array}{ccccc} \overline{B}(\mathbb{Z}[K\rtimes_\chi H]) & \overset{\varphi}{\cong} & C(\overline{W}(K\rtimes_\chi H)) & \overset{\psi}{\cong} & C(\overline{W}(K)\times_\tau \overline{W}(H)) \\ & & & & \Downarrow th.3 \\ hKH & \Leftarrow & \overline{B}(\mathbb{Z}[K])\otimes_t \overline{B}(\mathbb{Z}[H]) & \overset{\varphi^{-1}}{\cong} & C(\overline{W}(K))\otimes_t C(\overline{W}(H)) \end{array} $$

Composing the contractions above, we get

$$\phi\colon \overline{B}(\mathbb{Z}[K \rtimes_\chi G]) \underset{g}{\overset{f}{\rightleftharpoons}} hKH$$

i.e., a homological model for $K \rtimes_\chi H$.

Now we extend this homological model to the level of resolutions.

Theorem 7. *Suppose that $K \rtimes_\chi H$ is a semidirect product of finitely generated abelian groups. There exists a resolution $(\mathbb{Z}[K \rtimes_\chi H] \otimes hKH, d)$ which splits off of the bar resolution $B(\mathbb{Z}[K \times_\chi H])$.*

Proof. To Construct a resolution of the integers over $\mathbb{Z}[K \rtimes_\chi H]$ boils down to putting a $\mathbb{Z}[K \rtimes_\chi H]$-linear differential on $\mathbb{Z}[K \rtimes_\chi H] \otimes hKH$ such that an acyclic DG-module results. To this end, we follow these steps:

1. Construct the following contraction

$$1\otimes\phi\colon \mathbb{Z}[K \rtimes_\chi H] \otimes \overline{B}(\mathbb{Z}[K \rtimes_\chi H]) \underset{1\otimes g}{\overset{1\otimes f}{\rightleftharpoons}} (\mathbb{Z}[K \rtimes_\chi H] \otimes hKH,\ d^{\otimes}). \qquad (8)$$

2. Perturb the contraction above with $\theta\cap = d - d'$ where d is the differential on the bar resolution, $B(\mathbb{Z}[K \rtimes_\chi H]) = \mathbb{Z}[K \rtimes_\chi H] \otimes_\theta \overline{B}(\mathbb{Z}[K \rtimes_\chi H])$, and d' is the trivial differential on $\mathbb{Z}[K \rtimes_\chi H] \otimes \overline{B}(\mathbb{Z}[K \rtimes_\chi H])$. Obtaining:

$$(1\otimes\phi)_{\theta\cap}\colon \mathbb{Z}[K \rtimes_\chi H] \otimes_\theta \overline{B}(\mathbb{Z}[K \rtimes_\chi H]) \underset{(1\otimes g)_{\theta\cap}}{\overset{(1\otimes f)_{\theta\cap}}{\rightleftharpoons}} (\mathbb{Z}[K \rtimes_\chi H] \otimes hKH,\ d^{\otimes})$$

Obviously, $\epsilon\colon (\mathbb{Z}[K \rtimes_\chi H] \otimes hKH,\ d^{\otimes} + d_{\theta\cap}) \to \mathbb{Z}$ is the desired resolution and $d_{\theta\cap}$ is given explicitly by BPL.

Hence, we have to prove that the universal twisting cochain θ is a perturbation datum of (8). We organize the proof in three steps.

1. The contraction

$$0\colon \mathbb{Z}[K \rtimes_\chi H] \otimes C(\overline{W}(K \rtimes_\chi H)) \underset{1\otimes\varphi^{-1}}{\overset{1\otimes\varphi}{\rightleftharpoons}} \mathbb{Z}[K \rtimes_\chi H] \otimes C(\overline{W}(K) \times_\tau \overline{W}(H))$$

may be perturbed by means of the perturbation datum $\theta\cap$

$$\theta\cap((k,h)\otimes((k_1,h_1),\dots,(k_n,h_n))) = (k,h)\cdot(k_1,h_1)\otimes((k_2,h_2),\dots,(k_n,h_n)) - \\ -(k,h)\otimes((k_2,h_2),\dots,(k_n,h_n))$$

induced by the universal twisting cochain $\theta : \overline{B}(\mathbb{Z}[K \times_\chi H]) \to \mathbb{Z}[K \times_\chi H]$,

$$\theta([(k_1,h_1)|\cdots|(k_n,h_n)]) = \begin{cases} (k_1,h_1) - (e_K, e_H), & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

In fact, this step defines an isomorphism, since the homotopy operator is the zero map.

The perturbed differential $d_{\theta\cap}$ consists of

$$d_{\theta\cap}((k,h)\otimes(\{k_{n-1},\dots,k_0\},\{h_{n-1},\dots,h_0\})) =$$

$$((k,h)\cdot(h_{n-1}k_{n-1},h_{n-1}) - (k,h))\otimes(\{h_{n-1}k_{n-2},\dots,h_{n-1}k_0\},\{h_{n-2},\dots,h_0\}).$$

2. We now prove that $d_{\theta\cap}$ induces a finite perturbation process from

$$(1 \otimes AW_\delta,\, 1 \otimes EML_\delta,\, 1 \otimes SHI_\delta):$$

$$\mathbb{Z}[K \rtimes_\chi H] \otimes C(\overline{W}(K) \times_\tau \overline{W}(H)) \Rightarrow \mathbb{Z}[K \rtimes_\chi H] \otimes C(\overline{W}(K)) \otimes_t C(\overline{W}(H))$$

to

$$((1 \otimes AW_\delta)_{d_{\theta\cap}},\, (1 \otimes EML_\delta)_{d_{\theta\cap}},\, (1 \otimes SHI_\delta)_{d_{\theta\cap}}):$$

$$\mathbb{Z}[K \rtimes_\chi H] \otimes_\theta C(\overline{W}(K) \times_\tau \overline{W}(H)) \Rightarrow \mathbb{Z}[K \rtimes_\chi H] \tilde{\otimes} C(\overline{W}(K)) \otimes_t C(\overline{W}(H)).$$

Certainly, the map $(1 \otimes SHI_\delta) d_{\theta\cap}$ is pointwise nilpotent, as the filtration

$$F_q = \{(k,h) \otimes (\{k_{n-1}, \ldots, k_0\}, \{h_{n-1}, \ldots, h_0\}) : \\ \#(\{i :\, k_i = 0 \text{ or } h_i = 0\}) \geq n - q\}$$

shows. It is readily checked that $d_{\theta\cap}$ increases the filtration degree at most by 1 unit, since k_{n-1} and h_{n-1} cannot be simultaneously zero (we are working with normalized chain complexes). Taking into account that

$$SHI_\delta = \sum_{i \geq 0} (-1)^i [SHI((\delta\partial_0, \partial_0) - (\partial_0, \partial_0))]^i SHI,$$

it is evident that SHI_δ diminishes the filtration degree at least by 2 units, accordingly to the formulas for SHI (the filtration degree decreases by 2) and $SHI((\tau\partial_0, \partial_0) - (\partial_0, \partial_0))$ (the filtration degree decreases by 1).
An explicit formula for $\rho = d_{d_{\theta\cap}}$ is

$$\rho((k,h) \otimes \{k_{p-1}, \ldots, k_0\} \otimes \{h_{q-1}, \ldots, h_0\}) =$$

$$= ((k, hh_{q-1}) - (k,h)) \otimes \{h_{q-1}k_{p-1}, \ldots, h_{q-1}k_0\} \otimes \{h_{q-2}, \ldots, h_0\} +$$

$$+ ((k + hk_{p-1}, h) - (k,h)) \otimes \{k_{p-2}, \ldots, k_0\} \otimes \{h_{q-1}, \ldots, h_0\}.$$

3. Finally, the perturbation of the contraction

$$^{1\otimes\phi_t} \mathbb{Z}[K \rtimes_\chi H] \otimes C(\overline{W}(K)) \otimes_t C(\overline{W}(H)) \underset{1\otimes g_t}{\overset{1\otimes f_t}{\rightleftarrows}} \mathbb{Z}[K \times_\chi H] \otimes hK \otimes hH$$

by means of ρ converges, since $(1 \otimes \phi_t)\rho$ is pointwise nilpotent, as it may be concluded from the filtration

$$F_q = \{(k,h) \otimes \{k_{p-1}, \ldots, k_0\} \otimes \{h_{q-1}, \ldots, h_0\} : \sum_{i=0}^{p-1} |k_i| + \sum_{j=0}^{q-1} |h_j| \leq q\}. \quad \square$$

From this resolution, cohomology may be easily determined. It suffices to apply the *Hom* functor to the homological model just described. This is a classical procedure in Homological Algebra (e.g. see [21]).

In the setting of the simplicial groups we have an analogous result to Theorem 6 under certain hypothesis. More concretely, let us assume that K and H

are two simplicial groups where H operates on K from the left. Then we have the following chain of contractions:

$$C(\overline{W}(K \rtimes_\chi H)) \overset{\psi}{\cong} C(\overline{W}(K) \times_\tau \overline{W}(H)) \overset{th.3}{\Rightarrow} C(\overline{W}(K)) \otimes_t C(\overline{W}(H))$$

Furthermore, if two finite DG-modules hK and hH exist such that $C(\overline{W}(K)$ and $C(\overline{W}(H)$ contract to hK and hH, respectively, and the twisting cochain t (see 1) vanishes on simplicial degree 1 in $C(\overline{W}(H))$. Then the morphism $t\cap$ (see 2) is a perturbation datum of the contraction

$$C(\overline{W}(K) \otimes C(\overline{W}(H) \Rightarrow hK \otimes hH \tag{9}$$

(see [32, lemma 3.4.]). Hence, we can state the following theorm:

Theorem 8. *Under the circumstances displayed above. There exists a homological model for the semidirect product $K \rtimes_\chi G$ of simplicial groups H and K.*

Proof. This homological model is the composition of the following chain of contractions:

$$\begin{array}{ccccc} C(\overline{W}(K \rtimes_\chi H)) & \overset{\psi}{\cong} & C(\overline{W}(K) \times_\tau \overline{W}(H)) & \overset{th.3}{\Rightarrow} & C(\overline{W}(K)) \otimes_t C(\overline{W}(H)) \\ & & & & \Downarrow \\ & & & & (hK \otimes hH, d^{\otimes} + d_{t\cap}) \end{array}$$

The BPL yields the last contraction in the diagram above where the input data are the contraction (9) and the perturbation $t\cap$. □

Remark 3. If H is reduced, then the twisting cochain t (see 1) vanishes on simplicial degree 1 in $C(\overline{W}(H))$, as the following theorem states.

Theorem 9. *[34] Let $F \times_\tau B$ be a TCP with structural group G, and let e_0 denote the unit of G_0. If $\tau(b) = e_0, \quad \forall b \in B_1,$ then $t(b) = 0, \quad \forall b \in B_1$ where t denotes the cochain (1).*

3. Iterated semidirect products of finitely generated abelian groups

The definition of semidirect product of two groups G_1 and G_2 with respect to the homomorphism $\alpha\colon G_1 \to \mathrm{Aut}(G_2)$ denoted by $G_2 \rtimes_\alpha G_1$ can of course be iterated. Assume we are given groups $G_1, \ldots, G_l$ and, for each $1 < q \leq l$, homomorphisms

$$\alpha_q : G_{q-1} \to \mathrm{Aut}((\ldots(G_l \rtimes_{\alpha_l} G_{l-1}) \rtimes \ldots) \rtimes_{\alpha_{q+1}} G_q).$$

Then, we define the *iterated semidirect product* of $G_1, \ldots, G_l$ with respect to α_q to be the group

$$G = ((\ldots(G_l \rtimes_{\alpha_l} G_{l-1}) \rtimes \ldots) \rtimes_{\alpha_3} G_2) \rtimes_{\alpha_2} G_1).$$

In this section we extend the preceding work to the case of iterated semidirect products of finitely generated abelian groups.

Theorem 10. *Let G be an iterated semidirect product of finitely generated abelian groups. There exists a homological model* $^{\varphi:}\overline{B}(\mathbb{Z}[G]) \underset{g}{\overset{f}{\rightleftharpoons}} hG$ *for* G.

Proof. The filtrations used in the proof of Theorem 6 extend directly to this situation. □

Remark 4. Notice that the proof of the theorem above fits with iterated semidirect products of groups with G_i finitely generated abelian groups for $1 \leq i \leq l-1$, and with group G_l not necessarily abelian. This is the case of the iterated products of central extensions and semidirect products of finitely generated abelian groups considered in [2, 5].

Theorem 11. *Suppose that G is an iterated semidirect product of finitely generated abelian groups. There exists a resolution $(\mathbb{Z}[G] \otimes hG, d)$ which splits off of the bar resolution $B(\mathbb{Z}[G])$.*

Proof. Once again, the filtrations used in the proof of Theorem 7 extend in a straightforward way to this situation. □

4. Examples

All the executions and examples of this section have been worked out with the aid of the *Mathematica 4.0* notebook [5] described in [2], running on a *Pentium IV 2.400 Mhz DIMM DDR266 512 MB.*

Assume that

$$G = ((\ldots(G_l \rtimes_{\alpha_l} (G_{l-1}) \rtimes \ldots) \rtimes_{\alpha_3} G_2) \rtimes_{\alpha_2} G_1)$$

is an iterated semidirect product of finitely generated abelian groups. Each of the groups $G_i = \mathbb{Z}_{n_i}$ contributes a product $P_i = E(u_i) \otimes \Gamma(w_i)$ in the homological model hH, whereas each group $G_j = \mathbb{Z}$ contributes a single $P_j = E(u_j)$ in hH. The elements in $hH = \prod_{i=1}^{n} P_i$ are codified as linear combinations of tuples of length n, such that if m is the j^{th} entry of a tuple, it refers to $u_j^{m\ (mod\,2)} \otimes \gamma_{\lfloor \frac{m}{2} \rfloor}(w_j)$.

As the classical matrix algorithm indicates [46], in order to compute the homology $H_i(G)$ it is necessary to calculate Smith's normal forms of the matrices M_i and M_{i+1} to represent the differential operators d_i and d_{i+1}. The basis generators at degree i are ordered lexicographically, as follows

$$\{\{i, 0, \ldots, 0\}, \{i-1, 1, 0, \ldots, 0\}, \ldots, \{0, \ldots, 0, i\}\}.$$

We now include some calculations for dihedral groups and an iterated product of a central extension by a semidirect product of finite abelian groups. These groups have provided a large amount of cocyclic Hadamard matrices in [4, 7, 8], as well as some indispensable cohomological information for further theoretical research in [6].

4.1. Finite dihedral groups

$D_{4t} = \mathbb{Z}_{2t} \rtimes_\chi \mathbb{Z}_2$, $\chi : \mathbb{Z}_2 \times \mathbb{Z}_{2t} \to \mathbb{Z}_{2t}$ such that $\chi(1,x) = -x$ and $\chi(0,x) = x$. We next compute $H_i(D_{20})$ for $0 \le i \le 5$. We include the computing time which has been required in each case.

i	M_i	$H_i(D_{20})$	*Time*
0	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\mathbb{Z}$	$0''$
1	$\begin{pmatrix} 10 & 0 \\ -8 & 0 \\ 0 & 2 \end{pmatrix}$	$\mathbb{Z}_2^2$	$0.016''$
2	$\begin{pmatrix} 0 & 0 & 0 \\ 8 & 10 & 0 \\ -8 & -10 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\mathbb{Z}_2$	$0.156''$
3	$\begin{pmatrix} 10 & 0 & 0 & 0 \\ -80 & 0 & 0 & 0 \\ -360 & 10 & 10 & 0 \\ 288 & -8 & -8 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_{10}$	$2.46''$
4	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 80 & 10 & 0 & 0 & 0 \\ -656 & -82 & 0 & 0 & 0 \\ -2880 & -360 & 8 & 10 & 0 \\ 2304 & 288 & -8 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\mathbb{Z}_2^2$	$31.45''$
5	$\begin{pmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ -728 & 0 & 0 & 0 & 0 & 0 \\ -29520 & 82 & 10 & 0 & 0 & 0 \\ 238464 & -656 & -80 & 0 & 0 & 0 \\ 733440 & -2880 & -360 & 10 & 10 & 0 \\ -523776 & 2304 & 288 & -8 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$	$\mathbb{Z}_2^4$	$1h\ 1'\ 37''$

4.2. Infinite dihedral groups

$D_\infty = \mathbb{Z} \rtimes_\chi \mathbb{Z}_2$, $\chi : \mathbb{Z}_2 \times \mathbb{Z} \to \mathbb{Z}$ such that $\chi(1,x) = -x$ and $\chi(0,x) = x$.

We next compute $H_i(D_\infty)$. We include the computing time which has been required in each case.

i	M_i	$H_i(D_\infty)$	*Time*
0	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\mathbb{Z}$	$0''$
odd	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\mathbb{Z}_2^2$	$0.016''$
even > 0	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0	$0.015''$

4.3. An iterated product of finite groups

$G_t = (\mathbb{Z}_t \rtimes_f \mathbb{Z}_2) \rtimes_\chi \mathbb{Z}_2$, for χ being the dihedral action $\chi(a,b) = \begin{cases} -b & \text{if} \quad a=1 \\ b & \text{if} \quad a=0 \end{cases}$ and $f : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_t$ being the 2-cocycle $f(g_i,g_j) = \begin{cases} \lceil \frac{t}{2} \rceil + 1 & \text{if} \quad g_i = g_j = 1 \\ 0 & \text{otherwise} \end{cases}$.

Notice that $\mathbb{Z}_t \rtimes_f \mathbb{Z}_2$ is abelian (since f is symmetric), but G_t is not abelian for $t \neq 2$ (because of the dihedral action). Furthermore $G_t \simeq D_{4t}$ for odd t, since f is a 2-coboundary in these circumstances: $f = f_\alpha$, for $\alpha : \mathbb{Z}_2 \to \mathbb{Z}_t$ such that $\alpha(0) = 0$, $\alpha(1) = \dfrac{t^2+3}{4}$ *mod* t. Analogously, the extension is also trivial for $t \equiv 2$ *mod* 4, since $f = f_\alpha$, for $\alpha(0) = 0$, $\alpha(1) = \lfloor \dfrac{t}{4} \rfloor + 1$, so that $G_t \simeq (\mathbb{Z}_t \times \mathbb{Z}_2) \rtimes_\chi \mathbb{Z}_2$.

We next compute $H_i(G_4)$ for $0 \leq i \leq 4$. We include the computing time which has been required in each case.

i	M_i	$H_i(G_4)$	*Time*
0	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\mathbb{Z}$	$0''$
1	$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\mathbb{Z}_2^2$	$0.109''$
2	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ -2 & 0 & -4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\mathbb{Z}_2$	$0.594''$

i	M_i	$H_i(G_4)$	$Time$
3	$\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -2 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ -12 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 \\ -7 & -2 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\ 3 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_8$	$4.281''$
4	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -20 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 3 & 1 & 2 & 2 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -12 & -3 & 0 & -4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -24 & 0 & -12 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & -7 & -3 & -2 & 0 & 0 & 0 & -4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & -6 & -7 & 0 & -2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 6 & 0 & 0 & -2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & -3 & -4 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 8 & -2 & 3 & -1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ -2 & -1 & -2 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\mathbb{Z}_2^2$	$45.313''$

5. Related questions

We include here some comments about the simplifications of the formulas that our method provides.

In spite of the fact that a perturbation process is involved, the formulas for the morphisms EML_δ, AW_δ and SHI_δ (see (3)) in our method may be substantially reduced.

Proposition 12. *Consider the contraction*

$$^{SHI_\delta:}C(\overline{W}(K)\times_\tau\overline{W}(H))\underset{EML_\delta}{\overset{AW_\delta}{\rightleftarrows}}C(\overline{W}(K))\otimes_t C(\overline{W}(H)).$$

Then $SHI_\delta=SHI$, $EML_\delta=EML$ and $AW_\delta=AW-AW\delta SHI$.

PROOF. As we noted before (cf. Section 3), the perturbation datum associated to the perturbation process above, $\delta: C(\overline{W}(K)\times\overline{W}(H))\to C(\overline{W}(K)\times\overline{W}(H))$, consists of

$$\delta((k_{n-1},\dots,k_0),(h_{n-1},\dots,h_0))=((h_{n-1}k_{n-2},\dots,h_{n-1}k_0),(h_{n-2},\dots,h_0))-$$
$$-((k_{n-2},\dots,k_0),(h_{n-2},\dots,h_0)).$$

Defining $\overline{\kappa}: C(\overline{W}(K)\times\overline{W}(H))\to C(\overline{W}(K)\times\overline{W}(H))$ is given by

$$\overline{\kappa}((k_{n-1},\dots,k_0),(h_{n-1},\dots,h_0))=((h_{n-1}k_{n-1},\dots,h_{n-1}k_0),(h_{n-1},\dots,k_0)),$$

it is easily checked that $\delta=\partial_0\overline{\kappa}-\partial_0$, for $\partial_0=(\partial_0,\partial_0)$ being the degeneracy operator in $C(\overline{W}(K)\times\overline{W}(H))$.

It may be seen by inspection that an explicit formula for SHI consists of

$$SHI((k_{n-1},\dots,k_0),(h_{n-1},\dots,h_0))=\sum_{q=0}^{n-1}\sum_{p=0}^{n-q-1}\pm((k_{n-1},\dots,k_{p+q+1},0),$$
$$,(h_{n-1},\dots,h_{p+q+1},h_{p+q}\cdots h_q))||$$
$$||((k_{p+q},\dots,k_q),(1,\dots,1))*((0,\dots,0),(h_{q-1},\dots,h_0)),$$

where $*$ denotes the shuffle product and $||$ is used for juxtaposition.

This way,

$$SHI\overline{\kappa}((k_{n-1},\dots,k_0),(h_{n-1},\dots,h_0))=\overline{\kappa}SHI((k_{n-1},\dots,k_0),(h_{n-1},\dots,h_0))-$$
$$-\sum_{q=0}^{n-1}[(0,h_{n-1}\cdots h_q)||((h_{n-1}\cdots h_q a_{n-1},\dots,h_{n-1}\cdots h_q k_q),(1,\dots,1))*$$
$$*((0,\dots,0),(h_{q-1},\dots,h_0))+$$
$$+(0,h_{n-1}\cdots h_q)||((h_{n-1}k_{n-1},\dots,h_{n-1}k_q),(1,\dots,1))*((0,\dots,0),(h_{q-1},\dots,h_0))].$$

Furthermore,

$$\partial_0\overline{\kappa}SHI((k_{n-1},\dots,k_0),(h_{n-1},\dots,h_0))=\partial_0SHI\overline{\kappa}((k_{n-1},\dots,k_0),(h_{n-1},\dots,h_0))+$$
$$+\sum_{q=0}^{n-1}[((h_{n-1}\cdots h_q k_{n-1},\dots,h_{n-1}\cdots h_q k_q),(1,\dots,1))*((0,\dots,0),(h_{q-1},\dots,h_0))-$$
$$-((h_{n-1}k_{n-1},\dots,h_{n-1}k_q),(1,\dots,1))*((0,\dots,0),(h_{q-1},\dots,h_0))].$$

In these circumstances, since $EML\simeq -*-$, $SHI\ EML=0$ (from the side conditions of the Eilenberg-Zilber contraction) and $\partial_0SHI=-SHI\partial_0+EML\,AW$ (see [18] for the last relation), we may conclude that $SHI\delta SHI=0$:

- $SHI\partial_0 SHI = -SHI\,SHI\partial_0 + SHI\,EML\,AW = 0$.
- $SHI\partial_0\overline{\kappa}SHI = SHI\partial_0 SHI\overline{\kappa} + SHI\,EML(\,,\,) - SHI\,EML(\,,\,) = 0$.
- Thus $SHI\delta SHI = SHI\partial_0\overline{\kappa}SHI - SHI\partial_0 SHI = 0$.

Thus,

$$EML_\delta = \sum_{i\geq 0}(-1)^i(SHI\delta)^i EML = EML + SHI\delta EML.$$

Moreover, it is easy to check that

$$\delta EML = \partial_0\overline{\kappa}EML - \partial_0 EML = EML(\,,\,) - EML(\,,\,),$$

so that $SHI\delta EML = 0$ and $EML_\delta = EML$.

Analogously, $SHI_\delta = \sum_{i\geq 0}(-1)^i(SHI\delta)^i SHI = SHI$.

Finally, $AW_\delta = \sum_{i\geq 0}(-1)^i AW(\delta SHI)^i = AW - AW\delta SHI$, so that

$$AW_\delta((k_{n-1},\dots,k_0),(h_{n-1},\dots,h_0)) = (2AW - AW\overline{\kappa})((k_{n-1},\dots,k_0),(h_{n-1},\dots,h_0)) -$$

$$-\sum_{q=0}^{n-1}[((h_{n-1}\cdots h_q k_{n-1},\dots,h_{n-1}\cdots h_q k_q,0,\dots,0),(1,\dots,1,h_{q-1},\dots,h_0)) -$$

$$-((h_{n-1}k_{n-1},\dots,h_{n-1}k_q,0,\dots,0),(1,\dots,1,h_{q-1},\dots,h_0))].\ \Box$$

On the other hand, there is a deeper structure beneath the resolution of Theorem 7 (in terms of A_∞-structures), which may lead to future improvement of the formulas for the homological calculations.

Consider a simply connected DGA-coalgebra C (we will denote by Δ its coproduct) and a DG-module M. Attending to the results in [25, 23], one may always extend a contraction ${}^{\phi:}C \underset{g}{\overset{f}{\rightleftharpoons}} M$ to the Stasheff's cobar tilde construction [42],

$${}^{\overline{\Omega}(\phi):}\overline{\Omega}(C) \underset{\overline{\Omega}(g)}{\overset{\overline{\Omega}(f)}{\rightleftharpoons}} \tilde{\Omega}(M)$$

so that M is endowed a structure of A_∞-coalgebra derived from c (see [40, 41]).

We claim that the resolution of Theorem 7 fits into this situation, even though the coalgebra is not simply connected in this case.

Theorem 13. *The resolution of Theorem 7 is endowed with an A_∞-coalgebra structure, which is naturally inherited from the reduced complex of the standard bar resolution.*

Proof. We have to prove that the simplicial differential d_s on the cobar construction

$$d_s = \sum_{i=1}^{*} 1^{i-1}\otimes(s^{-1}\otimes s^{-1})\Delta s\otimes 1^{*-i}$$

is a perturbation datum of the following contraction

$$^{T^a(\phi)}: T^a(s^{-1}(\overline{B}(\mathbb{Z}[K \rtimes_\chi H])) \underset{T(g)}{\overset{T(f)}{\rightleftharpoons}} T^a(s^{-1}(hKH)).$$

(Where $T^a(M)$ denotes the *tensor algebra of a DG-module* M and $T(f)\colon T(M) \to T(N)$ is the morphism induced from $f\colon M \to N$. Finally, s and s^{-1} denote the suspension and desuspension functors, respectively.)

1. In the first stage, the perturbation of

   $$^{0}: T^a(s^{-1}(C(\overline{W}(K \rtimes_\chi H)))) \underset{T(\varphi^{-1})}{\overset{T(\varphi)}{\rightleftharpoons}} T^a(s^{-1}(C(\overline{W}(K) \times_\tau \overline{W}(H))))$$

   induces the contraction

   $$^{0}: \overline{\Omega}(C(\overline{W}(K \rtimes_\chi H))) \underset{T(\varphi^{-1})_{d_s}}{\overset{T(\varphi)_{d_s}}{\rightleftharpoons}} \overline{\Omega}(C(\overline{W}(K) \times_\tau \overline{W}(H))).$$

   The perturbed differential $\delta = T(\varphi) d_s T(\varphi^{-1})$ coincides with the simplicial differential on the cobar construction of $C(\overline{W}(K) \times_\tau \overline{W}(H)))$,

   $$\delta([\{(k^0_{r_0-1},\dots,k^0_0),(h^0_{r_0-1},\dots,h^0_0)\}|\cdots|\{(k^n_{r_n-1},\dots,k^n_0),(h^n_{r_n-1},\dots,h^n_0)\}]) =$$

   $$= \sum_{l=0}^{n-1}\sum_{j=1}^{r_l-1}(-1)^{r_0+\cdots+r_{l-1}-l-j}[\{(k^0_{r_0-1},\dots,k^0_0),(h^0_{r_0-1},\dots,h^0_0)\}|\cdots|$$
   $$|\{(h^l_{r_l-1},\dots,h^l_j),(h^l_{r_l-1},\dots,h^l_j)\}|$$
   $$|\{(h^l_j\cdots h_{r_l-1}k^l_{j-1},\dots,h^l_j\cdots h^l_{r_l-1}k^l_0),(h^l_{j-1},\dots,h^l_0)\}|\cdots|$$
   $$|\{(k^n_{r_n-1},\dots,k^n_0),(h^n_{r_n-1},\dots,h^n_0)\}].$$

2. The perturbation of

   $$^{T(SHI_\tau)}: T^a(s^{-1}(C(\overline{W}(K)\times_\tau\overline{W}(H)))) \underset{T(EML_\tau)}{\overset{T(AW_\tau)}{\rightleftharpoons}} T^a(s^{-1}(C(\overline{W}(K))\otimes_t C(\overline{W}(H))))$$

   by means of δ converges, as it may be concluded from the following filtration $\{F_q\}_{q\geq 0}$ on $D = T^a(s^{-1}(C(\overline{W}(K) \times_\tau \overline{W}(H))))$, such that an element

   $$[\{(k^0_{r_0-1},\dots,k^0_0),(h^0_{r_0-1},\dots,h^0_0)\}|\cdots|\{(k^n_{r_n-1},\dots,k^n_0),(h^n_{r_n-1},\dots,h^n_0)\}] \in D$$

   belongs to F_q if and only if the condition

   $$q \geq r_0+\cdots+r_n+\#(\{(k^l_j,h^l_j):\ k^l_j = 0 \text{ ó } h^l_j = 0,\ 0\leq j\leq r_l-1,\ 0\leq l\leq n\})$$

   is satisfied.
   Furthermore, the perturbed differential

   $$d_\delta = T(AW_\tau)\delta\sum_{i\geq 0}((-1)^i T(SHI_\tau)\delta)^i T(EML_\tau)$$

reduces to $T(AW)\delta T(EML)$, so that

$$d_\delta([\cdots|\{k^l_{r_l-1},\ldots,k^l_0\}\otimes\{h^l_{t_l-1},\ldots h^l_0\}|\cdots])=$$

$$=[\cdots|\{k^l_{r_l-1},\ldots,k^l_i\}\otimes\{h^l_{t_l-1},\ldots,h^l_j\}|\{k^l_{i-1},\ldots,k^l_0\}\otimes\{h^l_{j-1},\ldots,h^l_0\}|\cdots],$$

for $0<i<p$, $0<j<q$.

3. Finally we take d_δ as a perturbation datum for

$$^{T(\phi_t)}{:}T^a(s^{-1}(C(\overline{W}(K))\otimes_t C(\overline{W}(H)))\underset{T(g_t)}{\overset{T(f_t)}{\rightleftharpoons}}(T^a(s^{-1}(hKH)),d\otimes 1^*+\cdots+1^*\otimes d).$$

The finiteness of the perturbation process is guaranteed as soon as

$$T(\sum_{i\geq 0}(-1)^i((1\otimes\phi_H+\phi_K\otimes g_Hf_H)t\cap)^i(1\otimes\phi_H+\phi_K\otimes g_Hf_H))d_\delta$$

is shown to be pointwise nilpotent. To this end, it is enough to consider the filtration $\{F_q\}_{q\geq 0}$, where F_q is defined by those

$$[\{k^0_{r_0-1},\ldots,k^0_0\}\otimes\{h^0_{t_0-1},\ldots h^0_0\}|\cdots|\{k^n_{r_n-1},\ldots,k^n_0\}\otimes\{h^n_{t_n-1},\ldots h^n_0\}]$$

such that $\#(\{l:r_l>1\})+\#(\{j:t_j>1\})\leq q$. □

Finally, let us remark that in this paper we have not used spectral sequences. The perturbation machinery enables us to compute the higher differentials of the small complexes in a straightforward manner. An interesting question is to determine the relation between our approach (based on the perturbation technique) and the Lyndon-Hochschild-Serre spectral sequence in the case of semidirect products (see [9, 45]). The explicit combinatorial formulation of Steenrod squares given in [20] can be of help in order to cast new light on this subject.

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